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2002 J. Phys. A: Math. Gen. 35 1807

(http://iopscience.iop.org/0305-4470/35/7/402)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 1807–1812 PII: S0305-4470(02)29768-1

COMMENT

Reply to 'Lifshitz-point critical behaviour to $O(\epsilon_L^2)$'

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Received 4 October 2001, in final form 21 November 2001
Published 8 February 2002
Online at stacks.iop.org/JPhysA/35/1807

Abstract

We reply to a recent comment by Diehl and Shpot (2001 *J. Phys. A: Math. Gen.* **34** 9101) criticizing our paper (Albuquerque L C and Leite M M 2001 *J. Phys. A: Math. Gen.* **34** L327). We show that the approximation we use for evaluating higher loop integrals is consistent with homogeneity. A new renormalization group approach is presented in order to compare the two methods with high-precision numerical data concerning the uniaxial case. We stress that isotropic behaviour cannot be obtained from anisotropic behaviour.

PACS numbers: 75.40.−s, 75.40.Cx, 64.60.Kw

In a recent paper Diehl and Shpot [1] (DS) criticized a method proposed earlier in [2] to calculate the critical exponents ν_{L2}, η_{L2}, and γ_L at order $O(\epsilon_L^2)$ for systems presenting an m-fold Lifshitz point. Working entirely in momentum space we perform the calculations by using normalization conditions along with dimensional regularization. The symmetry point, used to define the renormalized theory, was chosen by setting the external momenta scale along the quartic (competing) directions equal to zero, while keeping the external momenta scale along the quadratic directions. A detailed account of this approach was given in the second paper of [2] for the uniaxial ($m = 1$) case.

In momentum space the most general solution to Feynman integrals involving quadratic and quartic external momenta scale subspaces perpendicular to each other is a difficult task, even at the one-loop level. Indeed, the one-loop integral I_2 contributing to the coupling constant can be performed exactly only if the external quartic momenta are set to zero. Keeping both external momenta scales, one can solve the integral over the quadratic momenta as a function of the external quadratic momenta, by choosing Schwinger parameters, for instance. The integral over the quartic momenta cannot be obtained as a function of the quartic external momenta in

a closed form. Setting the quartic external momenta to zero simplifies the problem, since this integral will contribute with a simple factor to the remaining parametric integrals, which can be solved in a straightforward manner. Absorbing a convenient geometric angular factor, the result can be cast in a form which resembles the ordinary ϕ^4 theory, with ϵ_L replacing ϵ and a slightly different coefficient for the regular term in ϵ_L. The result is a homogeneous function of the quadratic external momenta scale.

The parametric integrals play an interesting role in our approximation. To see this, consider the simplest two-loop integral contributing to the two-point function, namely $I_3(p, k')$ given by

$$I_3(p,k') = \int \frac{\mathrm{d}^{d-m}q_1\,\mathrm{d}^{d-m}q_2\,\mathrm{d}^m k_1\,\mathrm{d}^m k_2}{\left(q_1^2+\left(k_1^2\right)^2\right)\left(q_2^2+\left(k_2^2\right)^2\right)\left[(q_1+q_2+p)^2+\left((k_1+k_2+k')^2\right)^2\right]}. \tag{1}$$

Setting $k' = 0$, the integral can be evaluated as outlined in [2]. Before making our approximation, one can choose to integrate first either over the loop momenta (q_1, k_1) or over (q_2, k_2). The loop integrals to be integrated first are referred to as the internal bubbles. By solving the integral over q_2 first, we obtain

$$\begin{aligned} I_3(p,0) = {} & \frac{1}{2}S_{d-m}\Gamma\left(\frac{d-m}{2}\right)\int \frac{\mathrm{d}^{d-m}q_1\mathrm{d}^m k_1}{q_1^2+\left(k_1^2\right)^2} \\ & \times \int_0^\infty\int_0^\infty \mathrm{d}\alpha_1\mathrm{d}\alpha_2(\alpha_1+\alpha_2)^{\frac{-(d-m)}{2}}\exp\left(-\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}(q_1+p)^2\right) \\ & \times \int \mathrm{d}^m k_2 \mathrm{e}^{-\alpha_1\left(k_2^2\right)^2}\mathrm{e}^{-\alpha_2\left((k_1+k_2)^2\right)^2}. \end{aligned} \tag{2}$$

Now we can consider the approximation. In order to integrate over k_2, we have to expand the argument of the last exponential. This will produce a complicated function of α_1, α_2, k_1 and k_2. Unfortunately, this function has no elementary primitive. Considering the remaining terms as a damping factor to the integrand, the maximum of the integrand will be either at $k_1 = 0$ or $k_1 = -2k_2$. (The most general choice $k_1 = -\alpha k_2$ yields a hypergeometric function.) The choice $k_1 = -2k_2$ implies that k_1 varies in the internal bubble, but not arbitrarily. Its variation, however, is dominated by k_2 through this constraint, which eliminates the dependence on k_1 in the internal bubble. At these values of k_1, the integration over k_2 produces a simple factor to the parametric integral proportional to $(\alpha_1+\alpha_2)^{-\frac{m}{4}}$. This allows one to perform the remaining parametric integrals in a simple way. Thus, the constraint is designed to preserve the *form* of the parametric integrals. After realizing these integrals, they produce the factor $((q_1+p)^2)^{-\epsilon_L/2}$. Note that the diagrams I_3 and I_5 contributing to the two-point function receive the factor $\frac{1}{2-\frac{m}{4}}$ after integrating over the quadratic momenta in the external bubble. This factor will not be present in the isotropic case, since there is no integration over quadratic momenta to be done in this case. The resulting solution to $I_3(p, 0)$ is a homogeneous function of the external momenta p, not a generalized homogeneous function, given by

$$I_3 = -(p^2)^{1-\epsilon_L}\frac{1}{8-m}\frac{1}{\epsilon_L}\left[1+\left([\mathrm{i}_2]_m+\frac{3}{4-\frac{m}{2}}+1\right)\epsilon_L\right]. \tag{3}$$

The implementation of this constraint on higher loop integrals proceeds analogously. The constraint turns all these integrals into homogeneous functions of the external quadratic momenta scale. One can then choose the symmetry point as $p^2 = \kappa_1^2$, for example, in order to define the renormalized vertices via normalization conditions. The normalization constants $Z_\phi(\kappa_1)$, $Z_{\phi^2}(\kappa_1)$ and the beta function are defined in [2], which give origin to the exponents ν_{L2} and η_{L2} (under the momentum flow in the scale κ_1), along with all the scaling relations relative

to exponents perpendicular to the competing axes [3]. This follows in complete analogy to the usual ϕ^4 theory describing the Ising model. As the constraint is based on a physical principle (homogeneity), we do not agree that the approximation is unacceptable.

Actually, we agree that the speculation made in [2] ("... suggests that calculations performed in momentum space and coordinate space are inequivalent, as far as the Lifshitz point is concerned") was unfortunate. The DS study was the first one to extend the treatment for the $m = 2, 6$ cases to general m by making use of the scaling form of the free propagator in coordinate space representation. However, this does not give them support for their speculation that 'there is no way that ALs and our calculation can be both correct'. In the following we shall outline a new renormalization approach in momentum space of this problem using *two independent fixed points*, which is different from the DS treatment using only one fixed point and will help the subsequent discussion.

So far, we have obtained half the solution to the problem, as we still have to devise a method to calculate the critical exponents along the competing axes. We follow a suggestion made by Wilson in the early 1970s [4] in order to obtain these exponents independently of those perpendicular to the competition axes. We can consider another independent set of normalization conditions defined at zero-quadratic external momenta and non-vanishing *quartic* external momenta scale κ_2 [3]. At the Lifshitz point, the free propagator has only quartic momenta along the competition axes. Thus, it is possible to perform a dimensional redefinition of the m-dimensional subspace by considering the associated quartic momenta to have half the dimension of a conventional momentum scale. As a consequence, the term in the bare Lagrangian which is proportional to the quartic momenta does not need to be multiplied by another dimensionful normalization constant (σ_0) in order to be meaningful on dimensional grounds. Under a flow in κ_2 at the corresponding fixed point, the normalization constants $Z_\phi(\kappa_2)$, $Z_{\phi^2}(\kappa_2)$ lead to the critical exponents η_{L4}, ν_{L4} and new scaling laws along the competing axes, which are independent of those obtained in the subspace perpendicular to the competition directions. In this case, in order to evaluate loop integrals we use approximations which preserve the homogeneity of the Feynman integrals in the external quartic momenta scale κ_2, such that scaling theory is fulfilled. Specifically, consider the one-loop integral $I_2(0, P)$

$$I_2(0, P) = \int \frac{\mathrm{d}^{d-m}q\,\mathrm{d}^m k}{(((k+P)^2)^2+q^2)((k^2)^2+q^2)}. \tag{4}$$

The simplest approximation for this integral which preserves homogeneity in the quartic external momenta scale is $((k+P)^2)^2 = ((k)^2)^2 + ((P)^2)^2$. Of course, more involved approximations can be developed which preserve homogeneity, but we concentrate on this approximation for the sake of simplicity. The result for this integral is $I_2(0, P) = (((P)^2)^2)^{-\epsilon_L/2}\ \frac{1}{\epsilon_L}(1+[i_4]_m\epsilon_L)$, where the geometric angular factor $\frac{1}{2}\Gamma\left(\frac{m}{4}\right)\Gamma\left(2-\frac{m}{4}\right)S_{d-m}S_m$ has been absorbed in a redefinition of the coupling constant, and $[i_4]_m = \frac{1}{2}\left[1+\psi(1)-\psi\left(2-\frac{m}{4}\right)\right]$. This result reflects the independent infrared divergence of this integral on the external momenta scale along the competition axes. The beta function for this case, $\beta(u) = -2\epsilon_L\left(\frac{\partial \ln u_0}{\partial u}\right)_{\kappa_2}$, is different (and independent) from the one associated with critical exponents perpendicular to the competition axes, even though both have the same fixed point at one-loop level. It can be easily checked that at the one-loop $\nu_{L4} = \frac{\nu_{L2}}{2}$. Thus, homogeneity is the guiding principle for obtaining the solution to arbitrary loop integrals as a function of κ_2. The resulting scaling relations for the exponents associated with the correlations perpendicular to the competing axes are independent of those ones along the competition axes.

The renormalization group just described can be adapted to treat the isotropic behaviour $m = d$ close to 8. However, the isotropic case is intrinsically different from this renormalization

group perspective. There is only one momenta scale κ_2 and just one set of normalization conditions. The beta function $\beta(u) = -\epsilon_L \left(\frac{\partial \ln u_0}{\partial u}\right)_{\kappa_2}$ is half the value of the function associated with the κ_2 characterizing the competing directions in the anisotropic case. They are different, since the coupling constants in both cases have different canonical dimension. Technically, the isotropic loop integrals do not receive contributions from the parametric integration over the quadratic momenta subspace, for they are absent in this case. That is why the results of the anisotropic behaviour described in [2] cannot be extended to the isotropic one. The isotropic behaviour has its own scaling relations, which are independent of those concerning the correlations along the competition axes for the anisotropic case [3].

We can now analyse the previous RG formalisms possessing only one independent momenta scale for the anisotropic case. The first modern treatment in terms of 1PI vertex parts was given by Mergulhão and Carneiro [5]. There they set up the formalism in terms of normalization conditions in momentum space. They chose the symmetry point at non-vanishing quartic external momenta and zero quadratic external momenta as well as two conditions for the derivative of the two-point function at two independent external momenta scales. This reproduces the earlier scaling relations derived by Hornreich *et al* [6]. They went to coordinate space in order to calculate the exponents for the cases $m = 2, 6$. The novel feature of this approach is the introduction of an additional normalization constant σ_0, needed to obtain the exponents ν_{L4}, η_{L4}, etc. In [7], DS followed this treatment entirely in coordinate space in order to extend the formalism to the general m-fold behaviour. They introduced another normalization constant ρ_0 in order to treat the cross-over and identified the critical exponents using the renormalization group in coordinate space. The semi-analytical coefficients in the ϵ_L-expansion are integrals (generalized homogeneous functions) to be performed numerically in coordinate space. These numerical integrals only make sense if one splits the integration limits on the variable $\boldsymbol{v} = \sigma_0 \boldsymbol{x}_\parallel x_\perp$ using the scaling and related functions in the coordinate space representation in the integrand up to the maximum value of $|\boldsymbol{v}|$ at $|\boldsymbol{v}_0| = 9.3$, and replacing the asymptotic value of these functions for greater values of $\boldsymbol{v}$ [7]. This numerical approximation is needed in order to obtain a reasonable numerical value for the exponents η_{L2} and η_{L4} at $\mathrm{O}(\epsilon_L^2)$. Otherwise, the generalized homogeneous functions in the integrand are not suitable to describe properly the coefficients of the ϵ_L-expansion. They calculated I_2 and I_3 along these lines in [7].

After that, DS went to momentum space in order to calculate the two-loop integral $I_4(\boldsymbol{Q}, \boldsymbol{K})$ using dimensional regularization along with minimal subtraction [8]. They used a mixed treatment, calculating some integrals in momentum space, going to coordinate space whenever it was convenient (and vice versa) and making use of a scaling function Φ (defined in equation (8) of [8]). The integral I_2 is a subdiagram of I_4, depending on two external momenta scales as well. Nevertheless, they fixed the quartic external momenta scale to be zero and concluded that I_4 does not depend on it for general values of the quartic external momenta scale. Indeed, according to equation (B.14) of [8] $\left(\epsilon_L = 4 + \frac{m}{2} - d\right)$

$$I_4(Q\boldsymbol{e}_\perp, \boldsymbol{K}) = F^2_{m,\epsilon_L} \frac{Q^{-2\epsilon_L}}{2\epsilon_L} \left[\frac{1}{\epsilon_L} + J_u(m) + \mathrm{O}(\epsilon_L)\right]. \tag{5}$$

This happens to be incomplete. The problem can be traced back to the calculation of the one-loop integral

$$I_2(P_\perp, K'_\parallel) = \int \frac{\mathrm{d}^{d-m}q\,\mathrm{d}^m k}{((k+K')^4_\parallel + (q+P)^2_\perp)(k^4_\parallel + q^2_\perp)}. \tag{6}$$

They only computed this integral for vanishing external momenta along the quartic direction ($K'_{\|} = 0$). In this case, one has

$$I_2(P_\perp, K'_{\|} = 0) = (P^2)^{-\epsilon_L/2}\ I_2(\boldsymbol{e}_\perp). \tag{7}$$

However, by setting $P_\perp = 0$, and keeping $K'_{\|}$ different from zero, we have from the discussion following equation (4):

$$I_2(P_\perp = 0, K'\boldsymbol{e}_{\|}) = (K'^4)^{-\epsilon_L/2} I_2(\boldsymbol{e}_{\|}) \tag{8}$$

We stress that $I_2(\boldsymbol{e}_{\|})$ and $I_2(\boldsymbol{e}_\perp)$ are different functions in general. One can choose them to have the same leading singularities (the multiplicative factor is absorbed in a redefinition of the coupling constant anyway), the difference appearing in the regular terms in ϵ_L. In fact, the complete integral I_2 depends on these two momenta scales.

We recall that in a proper minimal subtraction procedure, *all* the external momenta should be kept *arbitrary* [9]. DS did not take into account this fact to proceed with the minimal subtraction, rather keeping only the quadratic external momenta $Q\boldsymbol{e}_\perp$ in I_4 and $P_\perp$ in I_2, and setting the quartic external momenta to zero in these integrals. They should show how the necessary cancellations of poles take place along the quartic subspace as well in order to have a satisfactory minimal subtraction scheme. In fact, the cancellations along the quadratic directions actually work when one fixes the quartic external momenta to zero. Even though this procedure is not complete, one can accept it as a new type of minimal subtraction for this problem.

For the anisotropic case, the DS method is based on one fixed point and the (almost exact) numerical integration for two-loop integrals which appear as the coefficients of the ϵ_L-expansion using coordinate space representations whenever it is convenient. On the other hand, the method developed in [3] (and discussed here) in momentum space utilizes two fixed points. This new method states that there are four independent critical exponents (instead of three) with two independent sets of scaling laws relating exponents along the quadratic and quartic directions in each subspace separately. We use an approximation, namely the constraint relating loop momenta in internal and external subdiagrams, which yields analytical results for higher loop integrals.

A comparison of the two methods with numerical results for the exponents associated with perpendicular correlations to the competing axes (labelled with the subscript L2 after [3]) is in order. For the $m = 1$, $d = 3$, $N = 1$ case, DS found using MATHEMATICA: $\nu_{L2} = 0.71$, $\gamma_{L2} = 1.40$. This is consistent with the newest Monte Carlo simulations for $\gamma_{L2} = 1.36 \pm 0.03$ [10], and compatible with an earlier Monte Carlo study (1.40 ± 0.06) [11]. On the other hand, our approximation yielded $\nu_{L2} = 0.73$ and $\gamma_{L2} = 1.45$ [2]. When using the new hyperscaling relation obtained in [3] for the specific heat exponent, namely $2 - \alpha_{L2} = \left(d - \frac{m}{2}\right)\nu_{L2}$ and replacing the value $\nu_{L2} = 0.73$, we obtain $\alpha_{L2} = 0.175$, while the most recent Monte Carlo calculation is $\alpha_{L2} = 0.18 \pm 0.02$ [10]. On the other hand, using the value obtained by DS $\nu_{L2} = 0.71$ in the new hyperscaling relation, we find $\alpha_{L2} = 0.225$. We can proceed and analyse the new scaling law obtained in [3] for the magnetization exponent $\beta_{L2} = \frac{1}{2}\nu_{L2}\left(\left(d - \frac{m}{2}\right) - 2 + \eta_{L2}\right)$. Our calculation yields $\beta_{L2} = 0.198$, whereas the simulation result is $\beta_{L2} = 0.238 \pm 0.005$. Using DS results for ν_{L2} and η_{L2} inside this new scaling relation one finds $\beta_{L2} = 0.192$. The high-precision numerical values [10] are in very good agreement with the two-loop results using our approximation, which we believe cannot be said to be 'unacceptable' at this point. The very similar values obtained for the exponents using either DS or our two-loop calculations confirms that momentum and coordinate space calculations should give the same results, since either our approximation or the DS numerical approximation is responsible for a rather small deviation in the two results when compared to the above numerical output.

Finally, we emphasize the failure of the DS method to treat the isotropic behaviour. As was pointed out in [2] and explained in this work, ν_{L2} and η_{L2} are not valid for the isotropic ($m = 8$) case. At the Lifshitz point, only the momenta scale along the competing axes is meaningful for the isotropic case. Hence one has to start from scratch using this momenta scale, which is incompatible with our choice of normalization conditions. DS calculated I_2 and I_4 along the components of quadratic external momenta only. But this momenta scale makes no sense for the $m = 8$ case at the Lifshitz point, since they are not present any longer. In that case the fixed point should be determined entirely as a function of the quartic external momenta scale as shown in [3]. There it was found that isotropic behaviour cannot be obtained from the anisotropic behaviour. This is in contradiction to DS and we conclude that it is most likely the use of the momentum scale vanishing at $m = 8$ that led DS to erroneous results.

In conclusion, we have shown that our two-loop results do constitute a very good approximation for calculating critical indices. Our method proved to be very simple to give analytical expressions to the exponents. It is based on a renormalization group analysis consisting of two independent fixed points and is a natural alternative to the DS semi-analytical approach based on only one fixed point. In view of the comparison with numerical values, we believe that both methods for the anisotropic cases deserve further investigation in order to unravel the fascinating issues concerning the Lifshitz critical behaviour.

Acknowledgments

The authors would like to thank B V Carlson for a critical reading of the manuscript and support from FAPESP, grant numbers 00/03277-3 (LCA) and 00/06572-6 (MML).

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